JOURNAL OF APPROXIMATION THEORY 47, 246–254 (1986)

Note on Operators of Szász–Mirakyan Type

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Communicated by Oved Shisha

Received February 8, 1985

1. INTRODUCTION

Let $C_A[0, +\infty)$ denote the set of functions $f \in C[0, +\infty)$ satisfying a growth condition of the form $|f(t) \leqslant Ae^{mt}$ $(A, m \in \mathbb{R}^+)$. Then, for $f \in C_A[0, +\infty)$, the well-known Szász–Mirakyan operator is defined by

$$S_n(f, x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \qquad (x \geqslant 0).$$

Replacing the infinite series by a finite partial sum, several authors also considered the operator

$$S_{n,N}(f; x) := e^{-nx} \sum_{k=0}^{N} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \qquad (x \geqslant 0)$$

for various choices of N. If, e.g., $N = N(n)$ is a sequence of positive integers with $\lim_{N \to \infty}(N(n)/n) = +\infty$, then Grof [5, p. 114] proves that

$$\lim_{n \to \infty} S_{n,N}(f; x) = f(x). \tag{1.1}$$

On the other hand, if $N = [n(x + \delta(n))]$ where $\lim_{n \to \infty} n^{1/2}\delta(n) = +\infty$, then Lehnhoff [6, Theorem 3] shows that (1.1) remains valid for $f \in C[0, +\infty)$ satisfying a growth condition of the form $|f(t)| \leqslant A + Bt^{2m}$ $(A, B \in \mathbb{R}^+, m \in \mathbb{N})$. In this note we show that Lehnhoff's result remains valid for all $f \in C_A[0, \infty)$ and all $N = N(n, x)$ satisfying

$$\lim_{n \to \infty} \frac{N - nx}{\sqrt{n}} = +\infty.$$

In case $\lim_{n \to \infty}((N - nx)/\sqrt{n}) = C$, a finite constant depending on x, we

0021-9045/86 $3.00

show that (1.1) no longer holds and should be modified. In this paper we also consider rates of convergence in (1.1) and in

$$\lim_{n\to\infty} S_n(f; x) = f(x). \tag{1.2}$$

Under appropriate conditions on f and N we will show that the rate of convergence in (1.1) and (1.2) is of the order $n^{-1/2}$. The results we give complement those of F. Cheng [2]. The method we use in proving our results depends heavily on the probabilistic interpretation of the operators $S_n(f; x)$ and $S_{n,N}(f; x)$, and therefore differs from the methods used by Lehnhoff or Cheng. It should be clear that our method extends easily to cover other operators of probabilitic type.

2. Main Results

Setting up our probabilistic argument, for $x \in \mathbb{R}^+$, let $X_1, X_2, \dots, X_n$ be independent random variables all having the same Poisson (x) distribution, i.e.,

$$P\{X_1 = k\} = e^{-x}\frac{x^k}{k!} \qquad (k = 0, 1, 2, \dots).$$

Now let $S_n = X_1 + X_2 + \cdots + X_n$; then S_n has a Poisson (nx) distribution and we obtain

$$S_n(f; x) = E\left(f\left(\frac{S_n}{n}\right)\right)$$

and

$$S_{n,N}(f; x) = E\left(f\left(\frac{S_n}{n}\right) I_{\{S_n \leqslant N\}}\right),$$

where $E(\cdot)$ denotes mathematical expectation and I_A denotes the indicator function of the set A. From probability theory we recall (see, e.g., [3, 4] or any other good book on probability theory)

(2.1) The Strong Law of Large Numbers:

$$\frac{S_n}{n} \to x \qquad (n \to \infty), \text{ almost surely;}$$

(2.2) Chebyshev's inequality: for every $t>0$ and $n \geqslant 1$

$$P\{|S_n - nx| > t\} \leqslant \frac{nx}{t^2};$$

(2.3) The Central Limit Theorem: for every $y \in \mathbb{R}$

$$P\left\{\frac{S_n - nx}{\sqrt{nx}} \leqslant y\right\} \to P\{Z \leqslant y\} = \Phi(y) := \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

(notation: $(S_n - nx)/\sqrt{nx} \Rightarrow^{\mathscr{D}} Z\ (n \to \infty)$);

(2.4) The Berry–Esseen theorem: for all $n \geqslant 1$

$$\sup_{y \in \mathbb{R}} \sqrt{n} \left| P\left\{\frac{S_n - nx}{\sqrt{nx}} \leqslant y\right\} - \Phi(y)\right| \leqslant C(x);$$

(2.5) A large deviation result: if y varies with n such that $y = o(n^{1/6})$ and $y \to \infty$, then

$$\frac{P\{(S_n - nx)/\sqrt{nx} \leqslant y\}}{P\{Z \leqslant y\}} \to 1 \qquad (n \to \infty).$$

Using these results we now prove the following classical result of Szász [7].

THEOREM 2.1. *For every $f \in C_A[0, \infty)$ we have*

$$\lim_{n \to \infty} S_n(f; x) = f(x)$$

uniformly on every interval $[x_1, x_2]$, $0 \leqslant x_1 < x_2 < \infty$.

Proof. First note that $E(e^{sS_n}) = e^{-nx + nxe^s} \leqslant e^{nxe^s}$ so that

$$E(e^{s(S_n/n)}) \leqslant e^{nxe^{s/n}} \qquad (s \geqslant 0). \tag{2.6}$$

Now since $f \in C_A[0, \infty)$ we obtain using (2.6)

$$|S_n(f; x)| \leqslant E\left|f\left(\frac{S_n}{n}\right)\right| \leqslant AE(e^{m(S_n/n)}) \leqslant Ae^{m'x} \tag{2.7}$$

for some $m' \in \mathbb{R}^+$. Also

$$E\left(\left(f\left(\frac{S_n}{n}\right)\right) - f(x)\right)^2\right) \leqslant A^2 e^{2m''x} \tag{2.8}$$

for some $m'' \in \mathbb{R}^+$. Now using the triangle inequality and then Schwarz' inequality [4, p. 152] we obtain: for all $\delta > 0$ and $x \geqslant 0$,

$$\begin{aligned}
|S_n(f; x) - f(x)| &\leqslant E\left(\left|f\left(\frac{S_n}{n}\right) - f(x)\right|\right) \\
&\leqslant E\left(\left|f\left(\frac{S_n}{n}\right) - f(x)\right| I_{\{|S_n/n - x| \leqslant \delta\}}\right) \\
&\quad + E\left(\left|f\left(\frac{S_n}{n}\right) - f(x)\right| I_{\{|S_n/n - x| > \delta\}}\right) \\
&\leqslant \sup_{\{y \geqslant 0 \,|\, |y - x| \leqslant \delta\}} |f(y) - f(x)| \\
&\quad + \left\{E\left(\left(f\left(\frac{S_n}{n}\right) - f(x)\right)^2\right) P\left\{\left|\frac{S_n}{n} - x\right| > \delta\right\}\right\}^{1/2}.
\end{aligned}$$

Using (2.2) with $t = \delta n$ and (2.8) we obtain for all $\delta > 0$ and $x \geqslant 0$ that

$$|S_n(f; x) - f(x)| \leqslant \sup_{\{y \geqslant 0 \,|\, |y - x| \leqslant \delta\}} |f(y) - f(x)| + Ae^{m''x} \sqrt{\frac{x}{n\delta^2}}. \quad (2.9)$$

Since f is uniformly continuous on every closed interval of $\mathbb{R}^+$ the desired result now follows from (2.9). ∎

To handle $S_{n,N}(f; x)$ note that since $I_A \leqslant 1$ we have, as in the proof of (2.9), that for $\delta > 0$ and $x \geqslant 0$,

$$\begin{aligned}
&|S_{n,N}(f; x) - f(x)\, P\{S_n \leqslant N\}| \\
&\quad \leqslant \sup_{\{y \geqslant 0 \,|\, |y - x| \leqslant \delta\}} |f(y) - f(x)| + Ae^{m''x} \sqrt{\frac{x}{n\delta^2}}.
\end{aligned} \quad (2.10)$$

Now we prove the following extension of Grof [5, p. 114] and Lehnhoff [6, p. 279].

THEOREM 2.2. (i) *If $N = N(n, x)$ is such that*

$$\lim_{n \to \infty} \frac{N - nx}{\sqrt{n}} = +\infty \quad (2.11)$$

then

$$\lim_{n \to \infty} S_{n,N}(f; x) = f(x). \quad (2.12)$$

(ii) *If (2.11) holds uniformly in $[x_1, x_2]$, $0 \leqslant x_1 < x_2 < \infty$, then also (2.12) holds uniformly in this interval.*

(iii) *If* $\lim_{n\to\infty}((N-nx)/\sqrt{n}) = C$, *a finite constant, then*

$$\lim_{n\to\infty} S_{n,N}(f;x) = f(x)\,\Phi\left(\frac{C}{\sqrt{x}}\right). \tag{2.13}$$

Proof. From (2.10) it follows that uniformly in $[x_1, x_2]$,

$$\lim_{n\to\infty} |S_{n,N}(f;x) - f(x)\,P\{S_n \leqslant N\}| = 0. \tag{2.14}$$

Now if (2.11) holds we have, using (2.3),

$$\lim_{n\to\infty} P\{S_n \leqslant N\} = \lim_{n\to\infty} P\left\{\frac{S_n - nx}{\sqrt{nx}} \leqslant \frac{N-nx}{\sqrt{nx}}\right\} = \Phi(\infty) = 1 \tag{2.15}$$

and (2.12) follows. If (2.11) holds uniformly in $[x_1, x_2]$ also (2.14), and hence (2.12), holds uniformly in $[x_1, x_2]$. Finally the proof of (iii) follows from (2.14) and

$$\lim_{n\to\infty} P\{S_n \leqslant N\} = \lim_{n\to\infty} P\left\{\frac{S_n - nx}{\sqrt{nx}} \leqslant \frac{N-nx}{\sqrt{nx}}\right\} = \Phi\left(\frac{C}{\sqrt{x}}\right). \quad \blacksquare$$

Our next result is devoted to the rate of convergence in (1.2). For a fixed $x > 0$ and $\delta > 0$ we will assume that $f \in C_A[0, \infty)$ and that

$$\left|\frac{f(t) - f(x)}{t - x}\right| \leqslant C(x, \delta) \qquad \text{for} \quad |t - x| \leqslant \delta,\ t \geqslant 0. \tag{2.16}$$

Here $C(x, \delta)$ denotes some constant depending on x and δ. Note that from (2.16) and $f \in C_A[0, \infty)$ we have

$$|f(t) - f(x)| \leqslant C'(x, \delta)\, e^{mt}\, |t - x|, \qquad t \geqslant 0 \tag{2.17}$$

for some constants C' and $m > 0$. Also note that (2.16) holds if $f'(x)$ exists. Now we prove

THEOREM 2.3. *If* $f \in C_A[0, \infty)$ *and if* (2.16) *holds, then*

$$\sup_{n\geqslant 1} \sqrt{n}\, |S_n(f;x) - f(x)| < \infty.$$

Furthermore, if $f'(x)$ *exists, then*

$$\lim_{n\to\infty} \sqrt{n}\,(S_n(f;x) - f(x)) = 0.$$

Proof. From (2.17), Schwarz' inequality, and (2.6) it follows that

$$\sqrt{n}\,|S_n(f;x)-f(x)| \leqslant \sqrt{n}\,E\left|f\left(\frac{S_n}{n}\right)-f(x)\right|$$
$$\leqslant C(x,\delta)\,E\left(e^{m(S_n/n)}\left|\frac{S_n-nx}{\sqrt{nx}}\right|\right)$$
$$\leqslant C''e^{m'x}\sqrt{E\left(\frac{S_n-nx}{\sqrt{nx}}\right)^2}=C''e^{m'x}$$

and the first result follows.

Next, suppose $f'(x)$ exists; since $S_n/n \to x$ almost surely ($n\to\infty$), we have

$$Z_n := \frac{f(S_n/n)-f(x)}{S_n/n-x}-f'(x)\to 0\ (n\to\infty),\ \text{almost surely.} \tag{2.18}$$

Now we have

$$\frac{f(S_n/n)-f(x)}{\sqrt{x/n}}=f'(x)\frac{S_n-nx}{\sqrt{nx}}+Z_n\frac{S_n-nx}{\sqrt{nx}}.$$

Using (2.18), (2.3), and [1, Theorem 4.1] we obtain

$$\frac{f(S_n/n)-f(x)}{\sqrt{x/n}}\overset{\mathscr{D}}{\Rightarrow} f'(x)Z. \tag{2.19}$$

Now from (2.17) and Schwarz' inequality we see that for $m>0$,

$$E\left(\left|\frac{f(S_n/n)-f(x)}{\sqrt{n/x}}\right|^m\right)\leqslant C''e^{m'x}\sqrt{E\left(\frac{S_n-nx}{\sqrt{nx}}\right)^{2m}}.$$

Using the boundedness of $E((S_n-nx)/\sqrt{nx})^{2m}$ (see, e.g., Lehnhoff [6, Lemma 4]) we obtain

$$\sup_{n\geqslant 1} E\left(\left|\frac{f(S_n/n)-f(x)}{\sqrt{x/n}}\right|^m\right)<\infty. \tag{2.20}$$

But then (2.19) and (2.20) together with [1, Theorem 5.4] imply that

$$E\left(\frac{f(S_n/n)-f(x)}{\sqrt{x/n}}\right)\to E(f'(x)Z)=0 \qquad (n\to\infty),$$

which proves the result. ∎

Remark. The example $f(t)=|t-x|$ shows that the first result of Theorem 2.3 is best possible. See also Cheng [2, p. 229].

Using the same method as in the proof of Theorem 2.3 we also have the following extension of the theorem.

THEOREM 2.4. *If* $f\in C_A[0,\infty)$ *and if* $x>0$, $\delta>0$, $k\in\mathbb{N}$ *are such that* $f^{(r)}(x)$ *exists for* $r=1, 2,..., k$ *and such that*

$$\left|f(t)-f(x)-\sum_{r=1}^{k}\frac{(t-x)^r}{r!}f^{(r)}(x)\right|\leqslant C(x,\delta)\,|t-x|^{k+1}$$

for $t\geqslant 0$, $|t-x|\leqslant\delta$, *then*

$$\sup_{n\geqslant 1} n^{(k+1)/2}\left|S_n(f;x)-f(x)-\sum_{r=2}^{k}\frac{E(S_n/n-x)^r}{r!}f^{(r)}(x)\right|<\infty.$$

Furthermore, if $f^{(k+1)}(x)$ *exists, then*

$$\lim_{n\to\infty} n^{(k+1)/2}\left(S_n(f;x)-f(x)-\sum_{r=2}^{k}\frac{E(S_n/n-x)^r}{r!}f^{(r)}(x)\right)$$
$$=f^{(k+1)}(x)\,x^{(k+2)/2}E(Z^{k+1}).$$

Remarks. 1. In view of the example $f(t)=e^t$, the result of Theorem 2.4 is best possible.

2. The constants $r_k := E(Z^{k+1})$ can be calculated more explicitly as

$$r_{2k}=0$$
$$r_{2k+1}=\frac{(2k+1)!}{k!2^{k+1}}\qquad (k=0, 1,...).$$

In our next theorem we obtain a rate of convergence result for the operators $S_{n,N}$.

THEOREM 2.5. *If* (2.14) *holds and if*

$$\liminf_{n\to\infty}\frac{N-nx}{\sqrt{n}\ln(n)}>0 \tag{2.21}$$

then

$$\sup_{n\geqslant 1}\sqrt{n}\,|S_{n,N}(f;x)-f(x)|<\infty. \tag{2.22}$$

Furthermore, if $f'(x)$ exists, if (2.21) *holds, and if*

$$\frac{N-nx}{\sqrt{n}}=o(n^{1/6}) \tag{2.23}$$

then

$$\lim_{n\to\infty}\sqrt{n}\,(S_{n,N}(f;x)-f(x))=0. \tag{2.24}$$

Proof. To prove (2.22) note that

$$\begin{aligned}&\sqrt{n}\,|S_{n,N}(f;x)-f(x)|\\&\quad\leqslant\sqrt{n}\,E\left|f\left(\frac{S_n}{n}\right)-f(x)\right|+\sqrt{n}\,|f(x)|\,P\{S_n>N\}.\end{aligned} \tag{2.25}$$

Now let $y=(N-nx)/\sqrt{nx}$; then $y\to\infty$ and $y>c\ln(n)$ for all n large. But then (2.4) implies that

$$\sqrt{n}\,P\{S_n>N\}\leqslant C(x)+\sqrt{n}(1-\Phi(y)).$$

Using $1-\Phi(y)\sim C'(e^{-y^2}/y)$ $(y\to\infty)$ and (2.21) we obtain that

$$\sup_{n\geqslant 1}\sqrt{n}\,P\{S_n>N\}<\infty. \tag{2.26}$$

The inequality (2.22) now follows from (2.25), (2.26), and Theorem 2.3.

To prove (2.24) note that

$$\begin{aligned}&\sqrt{n}\,(S_{n,N}(f;x)-f(x))\\&\quad=\sqrt{n}\,(S_n(f;x)-f(x))\,\sqrt{n}\,E\left(f\left(\frac{S_n}{n}\right)I_{\{S_n>N\}}\right).\end{aligned} \tag{2.27}$$

Using Schwarz' inequality and (2.7), we obtain

$$\sqrt{n}\left|E\left(f\left(\frac{S_n}{n}\right)I_{\{S_n>N\}}\right)\right|\leqslant C\sqrt{nP\{S_n>N\}}.$$

Now with y as before and using (2.5) we obtain

$$nP\{S_n>N\}\sim n(1-\Phi(y))\qquad(n\to\infty).$$

Using (2.21) and $1-\Phi(y)\sim C'(e^{-y^2}/y)$ $(y\to\infty)$ it follows that

$$nP\{S_n>N\}\to 0\qquad(n\to\infty). \tag{2.28}$$

Now (2.24) follows from (2.27), (2.28), and Theorem 2.3. ∎

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